UNIT 2

FURTHER ON WORKING WITH VARIABLES

Unit outcomes

After Completing this unit, you should be able to:

- solve life related problems using variables.
- multiply binomial by monomial and determine the product of binomials.
- determine highest common factor of algebraic expressions.

Introduction

By now you are well aware of the importance of variables in mathematics. In this unit you will learn more about variables, specially you will learn about mathematical expression, its component parts and uses of variables in formulas and solving problems. In addition to these you will study special expressions known as binomials and how to perform addition and multiplication on them.

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2.1. Further on Algebraic Terms and Expressions

2.1.1 Use of variables in formula

Group Work 2.1

Discuss with your friends

1. What a variable is?
2. Find what number I am left with if
   a. I start with x, double it and then subtract 6.
   b. I start with x, add 4 and then square the result.
   c. I start with x, take away 5, double the result and then divide by 3.
   d. I start with w, subtract x and then square the result.
   e. I start with n add p, cube the result and then divide by a.
3. Translate the following word problems in to mathematical expression.
   a. Eighteen subtracted from 3.
   b. The difference of -5 and 11.
   c. Negative thirteen subtracted from – 10.
   d. Twenty less than 32.
4. Describe each of the following sets using variables.
   a. The set of odd natural numbers.
   b. The solution set of $3x - 1 \geq 4$.
   c. The solution set of $x + 6 = 24$.
   d. The set of all natural number less than 10.

Definition 2.1: ***A variable*** **is a symbol or letter such as x, y and z used to represent an unknown number (value).**

Example 1: Describe each of the following sets using variables.

a. The solution set of $3x - 5 > 6$.

b. The solution set of $2x + 1 = 10$.

Solution

a) $3x - 5 > 6$ ……………. Given inequality

$3x - 5 + 5 > 6 + 5$ …….... Adding 5 from both sides

$3x > 11$ ………………… Simplifying

$\frac{3x}{3} > \frac{11}{3}$ ………………….. Dividing both sides by 3

$x > \frac{11}{3}$

The solution set of $3x - 5 > 6$ is $\left\{x : x > \frac{11}{3}\right\}$.

b) $2x + 1 = 10$ ………….. Given equation.

$2x + 1 - 1 = 10 - 1$ ……Subtracting 1 from both sides.

$2x = 9$ …….. ... Simplifying.

$\frac{2x}{2} = \frac{9}{2}$ ………… Dividing both sides by 2.

$x = \frac{9}{2}$

The solution set of $2x + 1 = 10$ is $x = \frac{9}{2}$ or S.S $= \left\{\frac{9}{2}\right\}$

Example 2: Find the perimeter of a rectangle in terms of its length ℓ and width w.

Solution Let P represent the perimeter of the rectangle.

Then P = AB + BC + CD + DA

$= \ell + w + \ell + w$

$= 2\ell + 2w$

$= 2(\ell + w)$

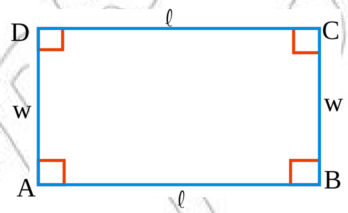

Figure 2.1

Example 3: The volume of a rectangular prism equals the product of the numbers which measures of the length, the width and the height. Formulate the statement using variables.

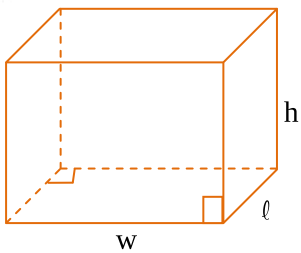

Figure 2.2 A rectangular prism

Solution let ℓ represent the length, w the width and h the height of the prism. If V represents the volume of the prism, then

$V = \ell \times w \times h$

$V = \ell wh$

Example 4: Express the area of a triangle in terms of its base 'b' and altitude 'h'.

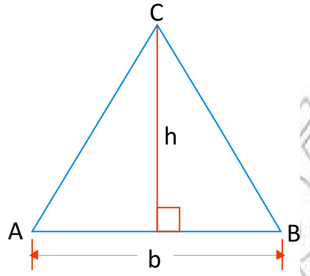

Figure 2.3 triangle

Solution Let "b" represent the base and "h" the height of the triangle.

$A = \frac{1}{2}bh$

Example 5: The area of a trapezium (see Figure 2.4) below can be given by the formula $A = \frac{1}{2}(b_1 + b_2)$ where A = area, h = height, b_1 = upper base and b_2 = lower base. If the area is 170 cm^2, height 17cm and b_2 = 12cm then:

a) Express b_1 in terms of the other variables in the formula for A.

b) Use the equation you obtained to find b_1.

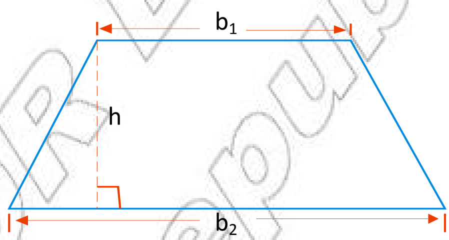

Figure 2.4 Trapezium

Solution

a) $\frac{1}{2}h(b_1 + b_2) = A$ …. Given equation

$h(b_1 + b_2) = 2A$ ….. Multiplying A by 2

$b_1 + b_2 = \frac{2A}{h}$ ….. Dividing both sides by h

$b_1 = \frac{2A}{h} - b_2$ ….. Subtracting b_2 from both sides

$b_1 = \frac{2A - b_2h}{h}$ …… Simplifying

b) For (a) above we have

$b_1 = \frac{2A - b_2h}{h}$

$b_1 = \frac{2(170) - 17(12)}{17}$

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$b_1 = \frac{17(20\text{-}12)}{17}$

$b_1 = 8\text{cm}$

Therefore the upper base (b_1) is 8cm.

✓ **Check:** $A = \frac{1}{2}h(b_1 + b_2)$ when $b_1 = 8\text{cm}$

$170\text{cm}^2 \stackrel{?}{=} \frac{1}{2}(17\text{cm})\ (8\text{cm} + 12\text{cm})$

$170\ \text{cm}^2 \stackrel{?}{=} \frac{17}{2}\ \text{cm}\ (20\text{cm})$

$170\ \text{cm}^2 = 170\ \text{cm}^2$ (True)

Exercise 2A

Solve each of the following word problems.

1. The perimeter of a rectangular field is 1000m. If the length is given as x, find the width in terms of x.
2. Find
   i) The perimeter of a square in terms of its side of length ″s″ unit.
   ii) The area of a square interms of its side of length ″s″ units.
3. Express the volume of the cube in Figure 2.5.

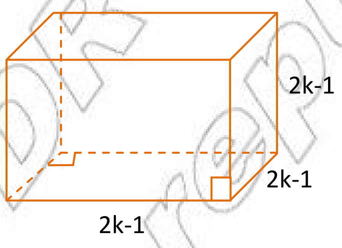

Figure 2.5 Cube

4. The area of a trapezoid is given by the formula $A = \left(\frac{b_1 + b_2}{2}\right)h$ then give the height h interms of its bases b_1 & b_2.
5. A man is 8x years old now. How old he will be in:
   a. 10 years time? b. 6x years time? c. 5y years time?
6. How many days (d) are there in the given number of weeks (w) below?
   a. 6 weeks c. y weeks
   b. 104 weeks d. 14 weeks

2.1.2 Variables, Terms and Expressions

Activity 2.1

Discuss with your teacher before starting the lesson.

1. What do we mean by like terms? Given an example.
2. Are $7a^3$, $5a^2$ and $12a$ like terms? Explain.
3. What is an algebraic expression?
4. What is a monomial?
5. What is a binomial?
6. What is a trinomial?
7. What are unlike terms? Give an example.

Definition 2.2: **Algebraic expressions are formed by using numbers, letters and the operations of addition, subtraction, multiplication, division, raising to power and taking roots.**

Some examples of algebraic expressions are:

$x + 10$, $y – 16$, $2x^2 + 5x – 8$, $x – 92$, $2x + 10$, etc.

Note:

i. An algebraic expression that contains variables is called an expression in certain variables. For examples the expression $7xy + 6z$ is an algebraic expression with variables x, y and z.
ii. An algebraic expression that contains no variable at all is called constant. For example, the algebraic expression $72 - 16\pi$ is constant.
iii. The terms of an algebraic expression are parts of the expression that are connected by plus or minus signs.

Examples 6: List the terms of the expression $5x^2 – 13x + 20$.

Solution: The terms of the expression $5x^2 – 13x + 20$ are $5x^2$, $- 13x$, and 20.

Definition 2.3: **An algebraic expression in algebra which contains one term is called a monomial.**

Example 7: $8x$, $13a^2b^2$, $\frac{-2}{3}$, $18xy$, $0.2a^3b^3$ are all monomial.

Definition 2.4: **An algebraic expression in algebra which contains two terms is called a binomial.**

Examples 8: $2x + 2y$, $2a - 3b$, $5p^2 + 8$, $3x^2 + 6$, $n^3 - 3$ are all binomial.

Definition 2.5: **An algebraic expression in algebra which contains three terms is called a trinomial.**

Examples 9: $4x^2 + 3x + 10$, $3x^2 - 5x + 2$, $ax^2 + bx + c$ are all trinomial.

Definition 2.6: Terms which have the same variables, with the corresponding variables are raised to the same powers are called like terms; other wise called unlike terms.

For example:

Like terms	Unlike terms
$34xy$ and $-8xy$	$12xy$ and $6x$ …… Different variables.
$18p^2q^3$ and p^2q^3	$8p^2q^3$ and $16p^3q^2$ …Different power
$5w$ and $6w$	$10w$ and 20 …… Different variables.
7 and 20	14 and $10a$…… Different variables.

Example 10: Which of the pairs are like terms: $80ab$ and $70b$ or $4c^2d^2$, and $-6c^2 d^2$.

Solution $4c^2d^2$ and $-6c^2d^2$ are like terms but $80\ ab$ and $70\ b$ are unlike terms.

Note:

i. Constant terms with out variables, (or all constant terms) are like terms.
ii. Only like terms can be added or subtracted to form a more simplified expression.
iii. Adding or subtracting like terms is called combining like terms.
iv. If an algebraic expression contains two or more like terms, these terms can be combined into a single term by using distributive property.

Example 11: Simplify by collecting like terms.

a. $18x + 27 - 6x - 2$
b. $18k - 10k - 12k + 16 + 7$

Solution

a. $18x + 27 - 6x - 2$
$= 18x - 6x + 27 - 2$ ……. Collecting like terms
$= 12x + 25$ ……………… Simplifying

b. $18k - 10k - 12k + 16 + 7$
$= 18k - 22k + 16 + 7$ ….. Collecting like terms
$= -4k + 23$ …….. ……… Simplifying

Example 12: Simplify the following expressions

a. $(6a + 9x) + (24a - 27x)$
b. $(10x + 15a) - (5x + 10y)$
c. $-(4x - 6y) - (3y + 5x) - 2x)$

Solution

a. $(6a + 9x) + (24a - 27x)$
$= 6a + 9x + 24a - 27x$ ……. Removing brackets
$= 6a + 24a + 9x - 27x$ ……. Collecting like terms
$= 30a - 18x$ ….. Simplifying

b. $(10x + 15a) - (5x + 10y)$

= 10x + 15a – 5x – 10y ……. Removing brackets
= 10x – 5x + 15a – 10y ……. Collecting like terms
= 5x + 15a – 10y …… . . . . . .Simplifying

c. –(4x – 6y) – (3y + 5x) – 2x
= -4x + 6y – 3y – 5x – 2x ……Removing brackets
= -4x – 5x – 2x + 6y – 3y ……Collecting like terms
= -11x + 3y ………………… Simplifying

Group work 2.2

1. Three rods A, B and C have lengths of (x + 1) cm; (x + 2) cm and (x – 3)cm respectively, as shown below.

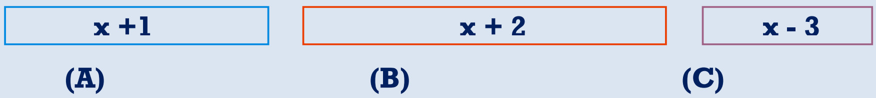

Figure 2.6

In the figures below express the length ℓ interms of x.

Give your answers in their simplest form.

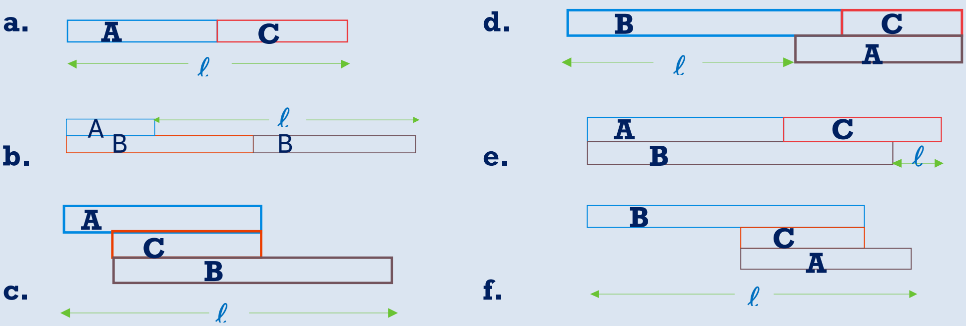

Figure 2.7

2. When an algebraic expression was simplified it became 2a + b.

a. Write down as many different expressions as you can which simplify to 2a + b.

b. What is the most complex expression you can think of that simplifies to 2a + b?

c. What is the simplest expression you can think of that simplifies to 2a + b?

Note: An algebraic formula uses letters to represent a relationship between quantities.

Exercise 2B

1. Explain why the terms $4x$ and $4x^2$ are not like terms.
2. Explain why the terms $14w^3$ and $14z^3$ are not like terms.
3. Categorize the following expressions as a monomial, a binomial or a trinomial.

a. 26	d. $16x^2$	g. $20w^4 - 10w^2$
b. $50\,bc^2$	e. $10a^2 + 5a$	h. $2t - 10t^4 - 10a$
c. $90 + x$	f. $27x + \frac{3}{2}$	i. $70z + 13z^2 - 16$

4. Work out the value of these algebraic expressions using the values given.
   a. $2(a + 3)$ if $a = 5$
   b. $4(x + y)$ if $x = 5$ and $y = -3$
   c. $\frac{7-x}{y}$ if $x = -3$ and $y = -2$
   d. $\frac{2a+b}{c}$ if $a = 3$, $b = 4$ and $c = 2$
   e. $2(b + c)^2 - 3(b - c)^2$ if $b = 8$ and $c = -4$
   f. $(a + b)^2 + (a + c)^2$ if $a = 2$, $b = 8$ and $c = -4$
   g. $c(a + b)^3$ if $a = 3$, $b = 5$ and $c = 40$

Challenge Problems

5. Solve for d if $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ if $x_1 = 3$, $y_1 = 4$ and $x_2 = 12$, $y_2 = 37$.
6. $y\,\frac{[3x+6y(x-20)]}{2x+12}$ if $x = 5$ and $y = \frac{1}{2}$
7. Collect like terms together.

a. $xy + ab - cd + 2xy - ab + dc$	d. $5 + 2y + 3y^2 - 8y - 6 + 2y^2 + 3$
b. $3x^2 + 4x + 6 - x^2 - 3x - 3$	e. $6x^2 - 7x + 8 - 3x^2 + 5x - 10$
c. $3y^2 - 6x + y^2 + x^2 + 7x + 4x^2$	f. $2x^2 - 3x + 8 + x^2 + 4x + 4$

2.1.3 Use of Variables to Solve Problems

Activity 2.2

Discuss with your teacher

1. Prove that the sum of two even numbers is an even numbers.
2. If the perimeter of a rectangle is 120cm and the length is 8cm more than the width, find the area.
3. The sum of three consecutive integers is 159. What are the integers?
4. The height of a ballon from the ground increases at a steady rate of x metres in t hours. How far will the ballon rise in n hours?

In this topic you are going to use variables to solve problems involving some unknown values and to prove a given statement.

? What is a proof?

A **proof** is an argument to show that a given statements is true. The argument depends on known facts, such as definitions, postulates and proved theorems.

Example13: (Application involving consecutive integers)

The sum of two consecutive odd integers is -188. Find the integers.

Solution: Let x represent the first odd integer, hence

x + 2 represents the second odd integer.

(First integer) + (Second integer) = total ……. Write an equation in words.

x + (x + 2) = -188 …… Write a mathematical equation

$x + x + 2 = -188$

$2x + 2 = -188$

$2x = -190$

$\frac{2x}{2} = \frac{-190}{2}$

$x = -95$

Therefore the integer are -95 and -93.

Example14: (Application involving Ages)

The ratio of present ages of a mother and her son is 12 : 5. The mother's age, at the time of birth of the son was 21 years. Find their present ages.

$\left(\text{Hint: } \frac{x}{y} = \frac{12}{5}\right)$

Solution:

Let x be the present age of the mother and y be that of her son.

Thus $x:y = 12:5$ or $\frac{x}{y} = \frac{12}{5}$

$5x - 12y = 0$ …... Equation 1

$x - y = 21$ …... Equation 2

From equation 2, we get $x = 21 + y$ …… Equation 3

Substituting equation 3 in to equation 1, we will get:

$5(21 + y) - 12y = 0$

$105 + 5y - 12y = 0$

$105 - 7y = 0$

$105 = 7y$

$y = 15$

Thus $x = 21 + y$

$x = 21 + 15 \Rightarrow x = 36$

Therefore, the present ages of a mother and her son are 36 years and 15 years respectively.

Exercise 2C

Solve each of the following word problems.

1. A 10 meter piece of wire is cut in to two pieces. One piece is 2 meters longer than the other. How long are the pieces?
2. The perimeter of a college basket ball court is 96 m and the length is 14m more than the width. What are the dimensions?
3. Ten times the smallest of three consecutive integers is twenty two more than three times the sum of the integers. Find the integers.
4. The surface area "S" of a sphere of radius r is given by the formula: $S = 4\pi r^2$.
   Find (i) the surface area of a sphere whose radius is 5 cm.
   (ii) the radius of a sphere whose surface area is $17\frac{1}{9}$ cm^2.
5. By what number must be 566 be divided so as to give a quotient 15 and remainder 11?
6. I thought of a number, doubled it, then added 3. The result multiplied by 4 came to 52. What was the number I thought of ?

7. One number is three times another, and four times the smaller added to five times the greater amounts to 133; find them.

Challenge Problems

8. If a certain number is increased by 5, one – half of the result is three – fifths of the excess of 61 over the number. Find the number.
9. Divide 54 in to two parts so that four times the greater equals five times the less.
10. Prove that the sum of any 5 consecutive natural numbers is divisible by 5.

2.2 Multiplication of Binomials

In grade seven you have studied about certain properties of multiplication and addition such as the commutative and associative properties of addition and multiplication and the distributive of multiplication over addition. In this sub–unit you will learn how to perform multiplication of monomial by binomial and multiplication of binomial by binomial.

2.2.1 Multiplication of Monomial by Binomial

Activity 2.3

Discuss with your friends /partners/

1. Multiply $4a$ by $2ab$	3. Multiply $6b$ by $(3a + 15b)$
2. Multiply $4b$ by $(2ab + 6b)$	4. Multiply $7ab$ by $(3ab - 6a)$

You begin this topic, let us look at some examples:

Example 15: Multiply $2x$ by $4yz$

Solution:

$$2x \times 4yz = 2 \times x \times 4 \times y \times z$$
$$= (2 \times 4)(x \times y \times z)$$
$$= 8\,xyz$$

Example 16: Multiply $4c^2$ by $(16\,abc - 5a^2\,bc)$

Solution:

$$4c^2 \times (16abc - 5a^2bc)$$
$$= (4c^2 \times 16abc) - (4c^2 \times 5a^2bc)$$
$$= (4 \times 16 \times c^2 \times c \times a \times b) - (4 \times 5 \times c^2 \times c \times a^2 \times b)$$
$$= 64\,c^3\,ab - 20\,c^3\,a^2b$$

Example 17: Multiply 4rt (5pq – 3pq)

Solution:

$4rt \times (5pq - 3pq)$
$= (4rt \times 5pq) - (4rt \times 3pq)$
$= (4 \times 5 \times r \times t \times p \times q) - (4 \times 3 \times r \times t \times p \times q)$
$= (20rt\,pq - 12\,rt\,pq)$
$= (20\text{-}12)\,rt\,pq$
$= 8pqrt$

? Do you recall the properties used in examples 15, 16, and 17 above?

1. Distributive properties

Group Work 2.3

1. In Figure 2.8 below, find the area of the shaded region.

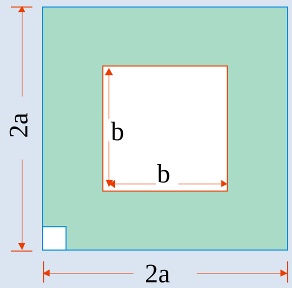

Figure 2.8

2. If the area of a rectangle is found by multiplying the length times the width, express the area of the rectangle in Figure 2.9 in two ways to illustrate the distributive property for a(b + c).

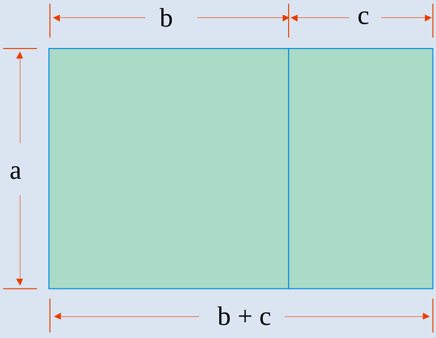

Figure 2.9

3. Express the shaded area of the rectangle in Figure 2.10 in two ways to illustrate the distributive property for a (b – c).

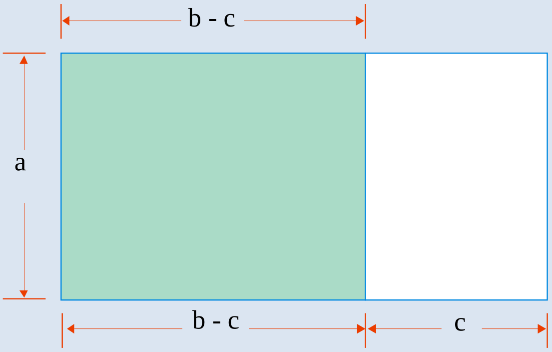

Figure 2.10

4. In Figure 2.11, find the area of each rectangle.

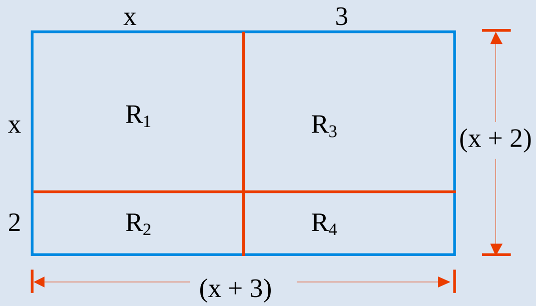

Figure 2.11

Consider the rectangle in Figure 2.12 which has been divided in to two smaller ones:

The area A of the bigger rectangle is given by: $A = a(b + c)$.

The area of the smaller rectangles are given by $A_1 = ab$ and $A_2 = ac$; but the sum of the areas of the two smaller rectangles are given by,

$A_1 + A_2$ gives the area A of the bigger rectangle:

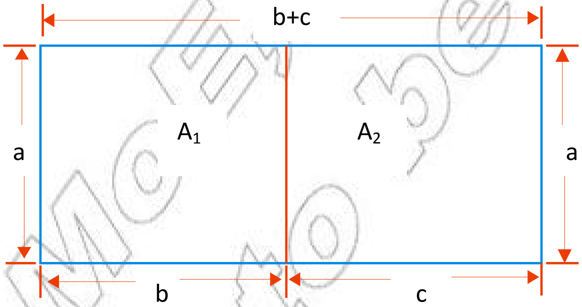

Figure 2.12 Rectangle

That means: $A = A_1 + A_2$

$a(b + c) = ab + ac$

This suggests that $a(b + c) = a \times b + a \times c$ the factor out side the bracket multiply each number in the bracket, this process of removing the bracket in a product is known as **expansion**.

Similarly consider another rectangle as in Figure 2.13 below.

Area of the shaded region = area of the bigger rectangle–area of the un shaded region.

Therefore, $a(b – c) = ab – ac$.

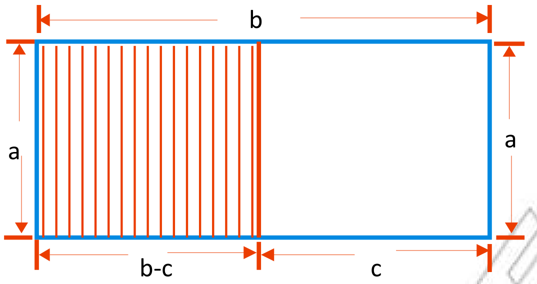

Figure 2.13 Rectangle

You have seen that the above two examples on area of rectangle, this could be generalized as in the following way:

Note: For any rational numbers a, b, and c

a. $a(b + c) = ab + ac$ b. $a(b – c) = ab – ac$

These two properties are called the distributive properties of multiplication over Addition (subtraction).

Exercise 2D

1. Expand these expressions by using the distributive properties to remove the brackets in and then simplify.

   1. $2(a + b) + 3(a + b)$
   2. $5(2a – b) + 49(a + b)$
   3. $4(5a+c) + 2(3a – c)$
   4. $5(4t – 3s) + 8(3t + 2s)$
   5. $5(3z + b) + 4(b – 2z)$
   6. $7(2d + 3e) + 6(2e – 2d)$
   7. $3(p + 2q) + 3(5p – 2q)$
   8. $5(5q + 4h) + 4(h – 5q)$
   9. $6(p + 2q + 3r) + 2(3p – 4q + 9r)$
   10. $2(a + 2b – 3c) + 3(5a – b + 4c) + 4(a + b + c)$

Challenge Problems

11. Remove the brackets and simplify.

a) $(x + 1)^2 + (x + 2)^2$
b) $(y - 3)^2 + (y - 4)^2$
c) $(x - 2)^2 + (x + 4)^2$
d) $(x + 2)^2 - (x - 4)^2$
e) $(2x + 1)^2 + (3x + 2)^2$
f) $(2x - 3)^2 + (5x + 4)^2$

2.2.2 Multiplication of Binomial by Binomial

Activity 2.4

Discuss with your friends / partners

Find the following products

1. $(2x + 8)(3x - 6)$
2. $(5a + 4)(4a + 6)$
3. $(2x - 8)(2x + 8)$
4. $(2x - 10)(x - 8)$
5. $(2a^2 - ab)(20 + x)$
6. $(3x^2 + 2x - 5)(x - 1)$

Sometimes you will need to multiply brackets expressions. For example $(a + b)(c + d)$. This means $(a + b)$ multiplied by $(c + d)$ or $(a + b) \times (c + d)$.

Look at the rectangles 2.14 below.

The area 'A' of the whole rectangle is $(a + b)(c + d)$. It is the same as the sum of the areas of the four rectangle so: $A = A_1 + A_2 + A_3 + A_4$

$$(a + b)(c + d) = ac + ad + bc + bd$$

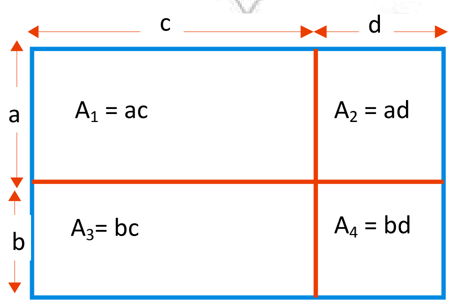

Figure 2.14 Rectangle

Notice that each term in the first brackets is multiplied by each term in the second brackets:

$a \times d$

$a \times c$

$(a + b)\ (c + d)$

$b \times c$

$b \times d$

You can also think of the area of the rectangle as the sum the areas of two separate part (the upper two rectangles plus the lower two rectangle) see Figure 2.15:

Thus, $(a + b)\ (c + d) = a\ (c + d) + b(c + d)$

Think of multiplying each term in the first bracket by the whole of the second bracket. These are two ways of thinking about the same process. The end result is the same. This is called **multiplying out** the brackets.

Figure 2.15 Rectangle

This process could be described as follows.

Note: If $(a + b)$ and $(c + d)$ are any two binomials their product $(a + b) \times (c + d)$ is defined as:

$$(a + b) \times (c + d) = ac + ad + bc + bd$$

Example18: Multiply (4x + 4) by (3x + 8)

Solution

$$(4x + 4)(3x + 8) = (4x \times 3x) + (4x \times 8) + (4 \times 3x) + (4 \times 8)$$
$$= 12x^2 + 32x + 12x + 32$$
$$= 12x^2 + 44x + 32$$

Example 19: Multiply (2x + 10) by (3x - 6)

Solution

$$(2x + 10)(3x - 6) = (2x \times 3x) - (6 \times 2x) + (10 \times 3x) - (6 \times 10)$$
$$= 6x^2 - 12x + 30x - 60$$
$$= 6x^2 + 18x - 60$$

Example20: Multiply (2x - 3) by (4x - 12)

Solution

$$(2x - 3)(4x - 12) = (2x \times 4x) - (12 \times 2x) - (3 \times 4x) + (3 \times 12)$$
$$= 8x^2 - 24x - 12x + 36$$
$$= 8x^2 - 36x + 36$$

In the multiplication of two binomials such as those shown in example 20 above, the product $2x \times 4x = 8x^2$ and $-3 \times -12 = 36$ are called **end products**. Similarly, the product $-12 \times 2x = -24x$ and $-3 \times 4x = -12x$ are called **cross product**. Thus the product of any two binomials could be defined as the sum of the **end products** and the **cross products**. The sum of the cross products is written in the middle.

Example21: Multiply (4x - 6) by (4x + 10)

Solution

$$(4x - 6)(4x + 10) = \overbrace{(4x \times 4x) + \underbrace{(4x \times 10) - (6 \times 4x)}_{\text{Cross product}} - (6 \times 10)}^{\text{end product}}$$

$$= 16x^2 + 40x - 24x - 60$$
$$= 16x^2 + 16x - 60$$

Example22: Multiply (4x-10) by (6x-2)

Solution:

end product

$$(4x - 10)(6x - 2) = (4x \times 6x) - (2 \times 4x) - (10 \times 6x) + (10 \times 2)$$

cross product

$$= 24x^2 - 8x - 60x + 20$$

$$= 24x^2 - 68x + 20$$

Exercise 2E

Find the products of the following binomials.

1. $(2x + 2y)(2x - 2y)$
2. $(3x + 16)(2x - 18)$
3. $(-4x - 6)(-20x + 10)$
4. $-5\,[(4x+y)(3x+2b)]$
5. $\left(\frac{3}{2} - \frac{2}{3}x\right)(2x + 1)$
6. $\left(\frac{x}{8} + \frac{x}{8}\right)\left(\frac{x}{8} - \frac{x}{4}\right)$
7. $\left(\frac{3}{2}x + \frac{4}{3}x\right)\left(\frac{2}{3}x + \frac{3}{5}x\right)$
8. $\left(\frac{4}{5}ab - \frac{3}{5}ab\right)\left(-4ab - \frac{3}{2}ab\right)$
9. $\left(\frac{2}{5}ab + \frac{3}{5}a^2b^2\right)\left(\frac{3}{7}a^2b^2 + \frac{3}{7}a^2b^2\right)$

Challenge Problems

10. $(2x^2 + 4x - 6)(x^2 + 4)$
11. $(2x^2 - 4x - 6)\left(\frac{3}{2}x^2 - 6\right)$
12. $(4x^2 + 4x - 10)(5x - 5)$

2.3 Highest Common Factors

Activity 2.5

Discuss with your teacher before starting the lesson.

1. Define and explain the following key terms:
   a. Factorizing a number
   b. Prime factorization
2. Find the HCF of the following.
   a. 72 and 220
   b. 36, 48 and 72
   c. 120, 150 and 200

3. Find the HCF of the following:
   a. $20xyz$ and $18x^2z^2$
   b. $5x^3y$ and $10xy^2$
   c. $3a^2b^2$, $6a^3b$ and $9a^3b^3$
4. Factorize the following expressions.
   a. $\frac{3}{19}ac - \frac{1}{19}ad$
   c. $\frac{5a^2b^2}{4} + \frac{15}{6}a^4b^2$
   b. $x(2b + 3) + y(2b + 3)$
   d. $a^2(c+2d)-b^2(c+2d)$

Factorizing

This unit is devoted to the method of describing an expression is called **Factorizing**. To factorize an integer means to write the integer as a product of two or more integers. To factorize a monomial or a Binomial means to express the monomial or Binomial as a product of two or more monomial or Binomials. In the product $2\times5 = 10$, for example, 2 and 5 are factors of 10. In the product $(3x + 4)(2x) = 6x^2 + 8x$, the expressions $(3x + 4)$ and $2x$ are factors of $6x^2 + 8x$.

Example 23: Factorize each monomial in to its linear factors with coefficient of prime numbers.

a. $15x^3$ b. $25x^3$

Solution:

a. $15x^3 = (3\times 5)\times(x\times x\times x)$
$= (3x)\times(5x)\times(x)$.

b. $25x^3 = (5\times 5)\times(x\times x\times x)$
$= (5x)\times(5x)(x)$.

Example 24: Factorize each of the following expressions.

a. $6x^2 + 12$ b. $5x^4 + 20x^3$

Solution:

a. $6x^2 + 12 = 6x^2 + 6\times 2$
$= 6(x^2 + 2)$

b. $5x^4 + 20x^3 = 5x^3\times x + 5x^3\times 4$
$= 5x^3(x + 4)$

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Exercise 2F

Factorize each of the following expressions.

1. $2x^2 + 6x$
2. $18xy^2 - 12xy^3$
3. $5x^3y + 10xy^2$
4. $16a^2b + 24ab^2$
5. $12ab^2c^3 + 16ac^4$
6. $3a^4b - 5bc^3$
7. $6x^4yz + 15x^3y^2z$
8. $8a^2b^3c^4 - 12a^3b^2c^3$
9. $8xy^2 + 28xyz - 4xy$
10. $-10mn^3 + 4m^2n - 6mn^2$

Challenge Problems

11. $7a^2b^3 + 5ab^2 + 3a^2b$
12. $2a^3b^3 + 3a^3b^2 + 4a^2b$
13. $-30abc + 24abc - 18a^2b$
14. $16x^4 - 24x^3 + 32x^2$
15. $10x^3 + 25x^2 + 15x$

Highest common factor of two integers

Group Work 2.4

Discuss with your group

For Exercise 1 – 4 factor out the HCF.

1. $15x^2 + 5x$
2. $5q^4 - 10q^5$
3. $y(5y + 1) - 9(5y + 1)$
4. $5x(x - 4) - 2(x - 4)$

You begin the study of factorization by factoring integers. The number 20 for example can be factored as 1×20, 2×10, 4×5 or $2\times2\times5$. The product $2\times2\times5$ (or equivalently $2^2\times5$) consists only of prime numbers and is called the **prime factorization**.

The **highest common factor** (denoted by HCF) of two or more integers is the highest factor common to each integer. To find the highest common factor of two integers, it is often helpful to express the numbers as a product of prime factors as shown in the next example.

Example 25: Find the highest common factor of each pair of integers.

a. 24 and 36

b. 105 and 40

Solution:

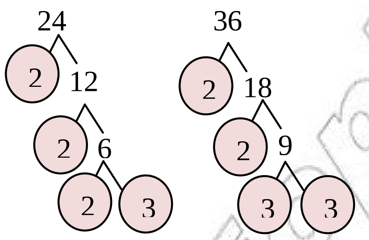
First find the prime factorization of each number by multiplication or by factor tree method.

a. i. by multiplication

Factors of 24 = $2 \times 2 \times 2 \times 3$

Factors of 36 = $2 \times 2 \times 3 \times 3$

ii. By using factor trees



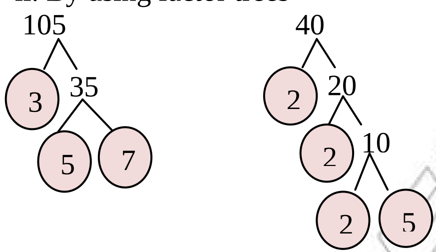
The numbers 24 and 36 share two factors of 2 and one factor of 3. Therefore, the highest common factor is $2\times2\times3 = 12$

b. i. By multiplication

Factors of 105 = $3 \times 7 \times 5$

Factors of 40 = $2\times2\times 2 \times 5$

ii. By using factor trees



Therefore, the highest common factors is 5.

Highest common factor (HCF) of two or more monomials

Example 26: Find the HCF among each group of terms.

a. $7x^3, 14x^2, 21x^4$

b. $8c^2d^7e, 6c^3d^4$

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Solution:

List the factors of each term.

a) $7x^3 = 7 \times x \times x \times x$

$14x^2 = 2 \times 7 \times x \times x$

$21x^4 = 3 \times 7 \times x \times x \times x \times x$

Therefore, the HCF is $7x^2$.

b) $\left.\begin{aligned} 8c^2d^7e &= 2^3c^2d^7e \\ 6c^3d^4 &= 2\times 3c^3d^4 \end{aligned}\right\}$ The common factors are the common powers of 2, c and d appearing in both factorization to determine the HCF we will take the common least powers. Thus

$\left.\begin{array}{l} \text{The lowest power of 2 is : } 2^1 \\ \text{The lowest power of c is : } c^2 \\ \text{The lowest power of d is : } d^4 \end{array}\right\}$

Therefore, the HCF is $2c^2d^4$.

Example 27: Find the highest common factor between the terms:

$3x(a + b)$ and $2y(a + b)$

Solution

$\left.\begin{array}{l} 3x\,(a + b) \\ 2y(a + b) \end{array}\right\}$ The only common factor is the binomial $(a + b)$.

Therefore, the HCF is $(a + b)$.

Factorizing out the highest common factor

Factorization process is the reverse of multiplication process. Both processes use the distributive property: $ab + ac = a\,(b + c)$

Example 28: Multiply: $5y(y^2 + 3y + 1)$

$= 5y\,(y^2) + 5y(3y) + 5y(1)$

$= 5y^3 + 15y^2 + 5y$

Factor: $5y^3 + 15y^2 + 5y$

$= 5y(y^2) + 5y(3y) + 5y(1)$

$= 5y\,(y^2 + 3y + 1)$

Example 29: Find the highest common factors

a. $6x^2 + 3x$ b. $15y^3 + 12y^4$ c. $9a^4b - 18a^5b + 27a^6b$

Solution a. The HCF of $6x^2 + 3x$ is $3x$ … Observe that $3x$ is a common factor.

$6x^2 + 3x = (3x \times 2x) + (3x \times 1)$ … Write each term as the product of $3x$ and another factor.

$= 3x\,(2x + 1)$ …… Use the distributive property to factor out the HCF.

Therefore, the HCF of $6x^2 + 3x$ is $3x$.

✓ **Check:** $3x\,(2x + 1) = 6x^2 + 3x$

b. The HCF of $15y^3 + 12y^4$ is $3y^3$ …Observe that $3y^3$ is a common factor.

$15y^3 + 12y^4 = (3y^3 \times 5) + (3y^3 \times 4y)$ …. Write each term as the product of $3y^3$ and another factor.

$= 3y^3(5 + 4y)$ ……Use the distributive property to factor out the HCF.

Therefore, the HCF of $15y^3 + 12y^4$ is $3y^3$.

c. $9a^4b - 18a^5b + 27a^6b$ is $9a^4b$ …Observe that $9a^4b$ is a common factor.

$= (9a^4b \times 1) - (9a^4b \times 2a) + (9a^4b \times 3a^2)$ …… Write each term as the product of $9a^4b$ and another factor.

$= 9a^4b(1 - 2a + 3a^2)$ …… Use the distributive property to factor out the HCF.

Therefore the HCF of $9a^4b - 18a^5b + 27a^6b$ is $9a^4b$.

Factorizing, out a binomial factor

The distributive property may also be used to factor out a common factor that consists of more than one term such as a binomial as shown in the next example.

Example 30: Factor out the highest common factor:
$2x(5x + 3) - 5(5x + 3)$

Solution

$2x(5x + 3) - 5(5x + 3)$ …….. The Highest common factor is the binomial $5x + 3$

$= (5x + 3) \times (2x) - (5x + 3) \times (5)$ ……… Write each term as the product of $(5x + 3)$ and another factor.

$= (5x + 3)(2x - 5)$ ….. Use the distributive property to factor out the HCF.

✓Check: $(5x + 3)(2x - 5) = (5x + 3)(2x) + (5x + 3)(-5)$

$= 2x(5x + 3) - 5(5x + 3)$

Exercise 2G

1. Find the highest common factor among each group of terms.
   a. $-8xy$ and $20y$
   b. $20xyz$ and $15yz^2$
   c. $6x$ and $3x^2$
   d. $2ab$, $6abc$ and $4a^2c$
   e. $3x^2y$, $6xy^2$ and $9xyz$
   f. $15a^3b^2$ and $20ab^3c$
   g. $6ab^4c^2$ and $12a^2b^3cd$
2. Find the highest common factor of the pairs of the terms given below.
   a. $(2a - b)$ and $3(2a - b)$
   b. $7(x - y)$ and $9(x - y)$
   c. $14(3x + 1)^2$ and $7(3x + 1)$
   d. $a^2(x + y)$ and $a^3(x + y)^2$
   e. $21x(x + 3)$ and $7x^2(x + 3)$
   f. $5y^3(y - 2)$ and $-20y(y - 2)$
3. Factor out the highest common factor.
   a. $13(a + 6) - 4b(a + 6)$
   b. $7(x^2 + 2) - y(x^2 + 2)$
   c. $8x(y^2 - 2) + (y^2 - 2)$
   d. $4(x + 5)^2 + 5x(x + 5) - (x + 5)$
   e. $6(z - 1)^3 + 7z(z - 1)^2 - (z - 1)$
   f. $x^4 - 4x$

Challenge Problems

4. Factor by grouping: $3ax + 12a + 2bx + 8b$

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Summary For Unit 2

1. *A **variable** is a symbol or letter used to represent an unspecified value in expression.*
2. *An **algebraic expression** is a collection of variables and constant under algebraic operations of addition or subtraction. For example, $y + 10$ and $2t - 2 \times 8$ are algebraic expressions.*

   The symbols used to show the four basic operations of addition, subtraction, multiplication and division are summarized in Table 2.1

Table 2.1

Operation	Symbols	Translation
Addition	$x + y$	✓ Sum of x and y ✓ x plus y ✓ y added to x ✓ y more than x ✓ x increased by y ✓ the total of x and y
Subtraction	$x - y$	✓ difference of x and y ✓ x minus y ✓ y subtracted from x ✓ x decreased by y ✓ y less than x
Multiplication	$x \times y$, $x(y)$, xy	✓ product of x and y ✓ x times y ✓ x multiplied by y
Division	$x \div y$, $\frac{x}{y}$, $^{x}/_{y}$	✓ Quotient of x and y ✓ x divided by y ✓ y divided into x ✓ ratio of x and y ✓ x over y

3. *For any rational numbers a, b and c*

   a. $a(b + c) = ab + ac$

   b. $a(b - c) = ab - ac$

   *These two properties are called the **distributive properties**.*

4. If $(a + b)$ and $(c + d)$ are any two binomials whose product $(a + b)(c + d)$ is defined as $(a + b)(c + d) = ac + ad + bc + bd$.

5. The highest common factor (HCF) of numbers is the greatest number which is a common factor of the numbers.

The procedures of one of the ways to find the HCF is given below:

1) List the factors of the numbers.

2) Find the common factors of the numbers.

3) Determine the highest common factors of these common factors.

Miscellaneous Exercise 2

I. State whether each statement is true or false for all positive integers x, y, z and w.

1. If a number y has z positive integer factors, then y and 2z integer factors.
2. If 2 is a factor of y and 3 is a factor of y, then 6 is a factor of y.
3. If y has exactly 2 positive integer factors, then y is a prime numbers.
4. If y has exactly 3 positive integer factors, then y is a square.
5. If y has exactly 4 positive integer factors, then y is a cube.
6. If x is a factor of y and y is a factor z, then x is a factor of z.

II. Choose the correct answer from the given four alternatives.

7. A triangle with sides 6, 8 and 10 has the same perimeter as a square with sides of length _______?

   a. 6 b. 4 c. 8 d. 12

8. If $x + y = 10$ and $x – y = 6$, what is the value of $x^3 – y^3$?

   a. 604 b. 504 c. 520 d. -520

9. If $ab + 5a + 3b + 15 = 24$ and $a + 3 = 6$, then $b + 5 =$ _____ ?

   a. 5 b. 50 c. 4 d. 12

10. If $ab = 5$ and $a^2 + b^2 = 25$, then $(a + b)^2 =$ ________ ?

    a. 35 b. 20 c. 15 d. 30

11. If n is an integer, what is the sum of the next three consecutive even integers greater than 2n?

    a. 6n + 12 b. 6n + 10 c. 6n + 4 d. 6n + 8

12. One of the following equation is false.

a. $A = \frac{1}{2}bh$ for $h = \frac{2A}{b}$

c. $P = 2(\ell + w)$ for $\ell = \frac{p}{2} - w$

b. $A = 2s^2 + 4sh$ for $h = \frac{A - 2s^2}{4s}$

d. $A = \frac{1}{2}bh$ for $h = \sqrt{\frac{2A}{h}}$

13. If x = 6 and y = 2, then what is the value of $3x^2 - 4\left(2y - \frac{4}{12}\right) + 8$.

a. $\frac{304}{3}$ b. $\frac{-304}{3}$ c. $\frac{-348}{3}$ d. $\frac{108}{3}$

14. Find the value of y, if $y = x^2 - 6$ and x = 7.

a. 49 b. 7 c. 43 d. 45

15. If x = 2 and y = 3, then what is the value of $y^x + xy \times y + x$?

a. 9 b. -29 c. 29 d. 18

16. If a = 4 and b = 7, then what is the value of $\frac{a + \frac{a}{b}}{a - \frac{a}{b}}$?

a. 8 b. 1 c. $\frac{4}{3}$ d. $\frac{8}{3}$

III. Work out Problems

17. Simplify each of the following expressions.

a. $(x^3 + 2x - 3) - (x^2 - 2x + 4)$

b. $2x(3x + 4) - 3(x + 5)$

c. $x(y^2 + 5xy) + 2xy(3x - 2y)$

d. $2(a^2b^2 - 4a^3b^3) - 8(ab^2 - 3a^2b^2)$

18. Express the volume of this cube.

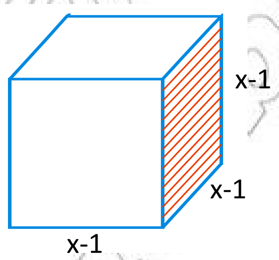

Figure 2.16 cube

19. Find the surface area of this cube.

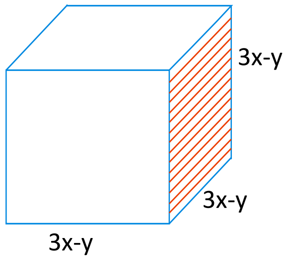

Figure 2.17 cube

20. Prove that the sum of five consecutive natural number is even.
21. Prove that $6(n + 6) – (2n + 3)$ is odd numbers for all $n \in \mathbb{N}$
22. Multiply the expressions.
    a. $(7x + y)(7x – y)$
    b. $(5k + 3t)(5k + 3t)$
    c. $(7x – 3y)(3x – 8y)$
    d. $(5z + 3)(z^2 + 4z – 1)$
    e. $\left(\frac{1}{3}m\text{-}n\right)^2$
    f. $(5a – 4b)(2a – b)$
    g. $\left(\frac{1}{5}x + 6\right)(5x – 3)$
    h. $(2h + 2{\cdot}7)(2h - 2{\cdot}7)$
    i. $(k – 3)^3$
    j. $(k + 3)^3$
23. Find the highest common factor for each expression.
    a. $12x^2 – 6x$
    b. $8x(x – 2) -2(x – 2)$
    c. $8(y + 5) + 9y(y + 5)$
    d. $y(5y + 1) – 8(5y + 1)$
    e. $4x(3x – y) + 5(3x – y)$
    f. $2(5x + 9) + 8x(5x + 9)$
    g. $8q^9 + 24q^3$
24. Find three consecutive numbers whose sum shall equal 45.
25. Find three consecutive numbers such that twice the greatest added to three times the least amount to 34.
26. Find two numbers whose sum is 36 and whose difference is 10.
27. If a is one factor of x, what is the other factor?
28. Find the value of $(x+5)(x+2) + (x–3)(x –4)$ in its simplest form. What is the numerical value when $x = -6$?
29. Simplify $(x + 2)(x + 10) – (x – 5)(x – 4)$. Find the numerical value of this expression when $x = -3$.

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